

C3 Differentiation

1a) $u = x^2$ $v = \cos 3x$
 $\frac{du}{dx} = 2x$ $\frac{dv}{dx} = -3 \sin(3x)$

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$= -3x^2 \sin(3x) + 2x \cos(3x)$$

b) $u = \ln(x^2 + 1)$ $v = (x^2 + 1)$
 $\frac{du}{dx} = \frac{2x}{(x^2 + 1)}$ $\frac{dv}{dx} = 2x$

$$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

$$= \frac{2x - 2x \ln(x^2 + 1)}{(x^2 + 1)^2}$$

c) $y = (4x + 1)^{1/2}$

$$\frac{dy}{dx} = 2(4x + 1)^{-1/2}$$

when $x = 2$ $\frac{dy}{dx} = \frac{2}{3}$

$$y = 3$$

$$y = \frac{2}{3}x + c \quad (2, 3)$$

$$3 = \frac{2}{3}(2) + c$$

$$3 = \frac{4}{3} + c$$

$$c = \frac{5}{3}$$

$$y = \frac{2}{3}x + \frac{5}{3}$$

$$3y = 2x + 5$$

$$\underline{0 = 2x - 3y + 5}$$

2a)

$$y = x^2 (5x - 1)^{1/2}$$

$$u = x^2$$

$$v = (5x - 1)^{1/2}$$

$$\frac{du}{dx} = 2x$$

$$\frac{dv}{dx} = \frac{5}{2}(5x - 1)^{-1/2}$$

$$\frac{dy}{dx} = \frac{5}{2}x^2(5x - 1)^{-1/2} + 2x(5x - 1)^{1/2}$$

when $x = 2$

$$\begin{aligned} \frac{dy}{dx} &= \frac{5}{2}(2)^2(5(2) - 1)^{-1/2} + 2(2)(5(2) - 1)^{1/2} \\ &= \frac{46}{3} \end{aligned}$$

b)

$$\frac{\sin 2x}{x^2}$$

$$u = \sin 2x$$

$$v = x^2$$

$$\frac{du}{dx} = 2 \cos 2x$$

$$\frac{dv}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{2x^2 \cos 2x - 2x \sin 2x}{x^4}$$

$$= \frac{2x \cos 2x - 2 \sin 2x}{x^3}$$

3a)

$$u = e^{3x}$$

$$v = \sin x + 2 \cos x$$

$$\frac{du}{dx} = 3e^{3x}$$

$$\frac{dv}{dx} = \cos x - 2 \sin x$$

$$\frac{dy}{dx} = e^{3x}(\cos x - 2 \sin x) + 3e^{3x}(\sin x + 2 \cos x)$$

$$= e^{3x}(\cos x - 2 \sin x + 3 \sin x + 6 \cos x)$$

$$= e^{3x}(7 \cos x + \sin x)$$

b/

$$y = \frac{3x^2 + 6x - 7}{(x+1)^2}$$

$$u = 3x^2 + 6x - 7 \quad v = (x+1)^2$$

$$\frac{du}{dx} = 6x + 6$$

$$\frac{dv}{dx} = 2(x+1)$$

$$\frac{dy}{dx} = \frac{(6x+6)(x+1)^2 - 2(x+1)(3x^2+6x-7)}{(x+1)^4}$$

$$= \frac{(6x+6)(x+1) - 2(3x^2+6x-7)}{(x+1)^3}$$

$$= \frac{6x^2 + 6x + 6x + 6 - 6x^2 - 12x + 14}{(x+1)^3}$$

$$= \frac{20}{(x+1)^3}$$

c/

$$\frac{dy}{dx} = 20(x+1)^{-3}$$

$$\frac{d^2y}{dx^2} = -60(x+1)^{-4}$$

$$-60(x+1)^{-4} = -\frac{15}{4}$$

$$\frac{-60}{(x+1)^4} = -\frac{15}{4}$$

$$(x+1)^4 = 16$$

$$x+1 = \pm 2$$

$$x = -1 \pm 2$$

$$\underline{x = 1} \quad \text{or} \quad \underline{x = -3}$$

4a)

$$y = e^{2x} \tan x$$

$$u = e^{2x}$$

$$v = \tan x$$

$$\frac{du}{dx} = 2e^{2x}$$

$$\frac{dv}{dx} = \sec^2 x$$

$$\frac{dy}{dx} = e^{2x} \sec^2 x + 2e^{2x} \tan x$$

turning points are where $\frac{dy}{dx} = 0$

$$e^{2x} \sec^2 x + 2e^{2x} \tan x = 0$$

$$e^{2x} (\sec^2 x + 2 \tan x) = 0$$

$$e^{2x} (\tan^2 x + 1) + 2 \tan x = 0$$

$$e^{2x} (\tan^2 x + 2 \tan x + 1) = 0$$

$$e^{2x} (\tan x + 1)(\tan x + 1) = 0$$

$$\underline{\tan x = -1} \quad \underline{\tan x = -1}$$

b) when $x=0$

$$\begin{aligned} \frac{dy}{dx} &= e^{2(0)} \sec^2(0) + 2e^{2(0)} \tan(0) \\ &= 1 \end{aligned}$$

when $x=0$, $y=0$

$$\underline{\underline{y=x}}$$

5a)

$$y = x^2 e^x$$

$$u = x^2$$

$$v = e^x$$

$$\frac{du}{dx} = 2x$$

$$\frac{dv}{dx} = e^x$$

$$\frac{dy}{dx} = x^2 e^x + 2x e^x$$

b) turning points where $\frac{dy}{dx} = 0$

$$x^2 e^x + 2x e^x = 0$$

$$e^x (x^2 + 2x) = 0$$

$$(e^x)(x)(x+2) = 0$$

$$\begin{aligned} x=0 & \quad x=-2 \\ (0,0) & \quad (-2, 4e^{-2}) \end{aligned}$$

$$c/ \quad \frac{dy}{dx} = x^2 e^x + 2x e^x$$

$$u = x^2 \quad v = e^x \quad u = 2x \quad v = e^x$$

$$\frac{d^2y}{dx^2} \quad \frac{du}{dx} = 2x \quad \frac{dv}{dx} = e^x \quad \frac{du}{dx} = 2 \quad \frac{dv}{dx} = e^x$$

$$\frac{d^2y}{dx^2} = x^2 e^x + 2x e^x + 2x e^x + 2e^x$$

$$= x^2 e^x + 4x e^x + 2e^x$$

$$d/ \quad x=0 \quad (0)^2 e^0 + 4(0)e^0 + 2e^0 = 2$$

+ve \therefore minimum

$$x=-2 \quad (-2)^2 e^{-2} + 4(-2)e^{-2} + 2e^{-2} = -2e^{-2}$$

-ve \therefore maximum

$$6. \quad x = 2 \sin y$$

$$\sqrt{2} = 2 \sin \frac{\pi}{4}$$

$$\sqrt{2} = 2 \frac{\sqrt{2}}{2}$$

$$\sqrt{2} = \sqrt{2} \quad \text{shown.}$$

$$b/ \quad \frac{dx}{dy} = 2 \sin y \cos y$$

$$\text{when } y = \frac{\pi}{4}$$

$$\frac{dx}{dy} = 2 \cos\left(\frac{\pi}{4}\right)$$

$$= 2 \frac{\sqrt{2}}{2}$$

$$= \sqrt{2}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{2}}$$

c/ $m = -\sqrt{2}$

$$y = -\sqrt{2}x + c \quad \begin{matrix} (\sqrt{2}, \pi/4) \\ x \quad y \end{matrix}$$

$$\pi/4 = -\sqrt{2}(\sqrt{2}) + c$$

$$\pi/4 = -2 + c$$

$$c = \pi/4 + 2$$

$$y = -\sqrt{2}x + \pi/4 + 2$$

7/1 $y = \frac{x}{9+x^2}$

$$u = x \quad v = 9+x^2$$

$$\frac{du}{dx} = 1 \quad \frac{dv}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{9+x^2 - 2x^2}{(9+x^2)^2} = \frac{9-x^2}{(9+x^2)^2}$$

turning points where $\frac{dy}{dx} = 0$

$$\frac{9+x^2 - 2x^2}{(9+x^2)^2} = 0$$

$$9+x^2 - 2x^2 = 0$$

$$9 = x^2$$

$$x = \pm 3$$

$$y = \pm 1/6$$

ü/ $y = (1 + e^{2x})^{3/2}$

$$\frac{dy}{dx} = \frac{3}{2} 2e^{2x} (1 + e^{2x})^{1/2}$$

$$= 3e^{2x} (1 + e^{2x})^{1/2}$$

when $x = \frac{1}{2} \ln 3$

$$\frac{dy}{dx} = 18$$

$$8 \quad y = e^{3x} + \ln 2x$$

$$\frac{dy}{dx} = 3e^{3x} + \frac{1}{x}$$

$$b) \quad y = (5+x^2)^{3/2}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{3}{2} \cdot 2x (5+x^2)^{1/2} \\ &= 3x (5+x^2)^{1/2} \end{aligned}$$

$$9 \quad y = \ln\left(\frac{1}{3}x\right)$$

$$\frac{dy}{dx} = \frac{1}{x} \quad \text{when } x=3, \quad y=0, \quad m = \frac{1}{3}$$

$$y = \frac{1}{3}x + c$$

$$0 = \frac{1}{3}(3) + c$$

$$c = -1$$

$$y = \frac{1}{3}x - 1$$

$$10a) \quad i) \quad u = x^2 \quad v = e^{3x+2}$$

$$\frac{du}{dx} = 2x \quad \frac{dv}{dx} = 3e^{3x+2}$$

$$\frac{dy}{dx} = \frac{\partial}{\partial u} e^{3x+2} + 2x e^{3x+2}$$

$$ii) \quad \frac{\cos(2x^3)}{3x}$$

$$u = \cos(2x^3) \quad v = 3x$$

$$\frac{du}{dx} = -6x \sin(2x^3) \quad \frac{dv}{dx} = 3$$

$$\frac{dy}{dx} = \frac{-18x^2 \sin(2x^3) - 3 \cos(2x^3)}{(3x)^2}$$

$$b) \quad x = 4 \sin(2y + 6)$$

$$\frac{dx}{dy} = 8 \cos(2y + 6)$$

$$\frac{dy}{dx} = \frac{1}{8 \cos(2y + 6)}$$

$$\left. \begin{aligned} \sin^2 x + \cos^2 x &= 1 \\ \cos^2 x &= 1 - \sin^2 x \\ \cos x &= \sqrt{1 - \sin^2 x} \end{aligned} \right\}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{1}{8\sqrt{1 - \sin^2(2y+6)}} \\ &= \frac{1}{8\sqrt{1 - \left(\frac{x}{4}\right)^2}} \end{aligned}$$

$$11a) \quad y = 3 \sin^2 x + \sec 2x$$

$$\frac{dy}{dx} = 6 \sin x \cos x + 2 \sec 2x \tan 2x$$

$$ii) \quad y = (x + \ln 2x)^3$$

$$\frac{dy}{dx} = 3(x + \ln 2x)^2 \left(1 + \frac{1}{x}\right)$$

$$b) \quad y = \frac{5x^2 - 10x + 9}{(x-1)^2}$$

$$u = 5x^2 - 10x + 9$$

$$v = (x-1)^2$$

$$\frac{du}{dx} = 10x - 10$$

$$\frac{dv}{dx} = 2(x-1)$$

$$\frac{dy}{dx} = \frac{(10x-10)(x-1)^2 - 2(x-1)(5x^2-10x+9)}{(x-1)^4}$$

$$= \frac{(10x-10)(x-1) - 2(5x^2-10x+9)}{(x-1)^3}$$

$$\frac{10x^2 - 10x - 10x + 10 - 10x^2 + 20x - 18}{(x-1)^3}$$

$$\frac{-8}{(x-1)^3}$$

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$$f(x) = 1 - \frac{2}{(x+4)} + \frac{x-8}{(x-2)(x+4)}$$

$$= \frac{(x-2)(x+4)}{(x-2)(x+4)} - \frac{2(x-2)}{(x-2)(x+4)} + \frac{x-8}{(x-2)(x+4)}$$

$$= \frac{x^2 + 4x - 2x - 8 - 2x + 4 + x - 8}{(x-2)(x+4)}$$

$$= \frac{x^2 + x - 12}{(x-2)(x+4)}$$

$$= \frac{(x+4)(x-3)}{(x-2)(x+4)}$$

$$= \frac{x-3}{x-2}$$

b)

$$\frac{e^x - 3}{e^x - 2}$$

$$u = e^x - 3 \quad v = e^x - 2$$

$$\frac{du}{dx} = e^x$$

$$\frac{dv}{dx} = e^x$$

$$\frac{dy}{dx} = \frac{e^x(e^x - 2) - e^x(e^x - 3)}{(e^x - 2)^2}$$

$$= \frac{e^{2x} - 2e^x - e^{2x} + 3e^x}{(e^x - 2)^2}$$

$$= \frac{e^x}{(e^x - 2)^2}$$

$$e) \frac{e^x}{(e^x - 2)^2} = 1$$

$$e^x = (e^x - 2)^2$$

$$e^x = e^{2x} - 4e^x + 4$$

$$0 = e^{2x} - 5e^x + 4$$

$$0 = (e^x - 4)(e^x - 1)$$

$$e^x = 4 \quad e^x = 1$$

$$\underline{x = \ln 4} \quad \underline{x = 0}$$

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$$\frac{2x+2}{x^2-2x-3} - \frac{x+1}{x-3}$$

$$\frac{2x+2}{(x-3)(x+1)} - \frac{(x+1)^2}{(x-3)(x+1)}$$

$$\frac{2x+2 - (x^2+2x+1)}{(x-3)(x+1)}$$

$$\frac{2x+2 - x^2 - 2x - 1}{(x-3)(x+1)}$$

$$\frac{1-x^2}{(x-3)(x+1)}$$

$$\frac{(1+x)(1-x)}{(x-3)(x+1)}$$

$$\frac{1-x}{x-3}$$

b)

$$u = 1-x \quad v = x-3$$

$$\frac{du}{dx} = -1 \quad \frac{dv}{dx} = 1$$

$$\frac{-1(x-3) - (1-x)}{(x-3)^2}$$

$$\frac{-x+3 - 1+x}{(x-3)^2}$$

$$\frac{2}{(x-3)^2}$$

14a)

$$\frac{2x+3}{x+2} - \frac{9+2x}{2x^2+3x-2}$$

$$\frac{2x+3}{x+2} - \frac{9+2x}{(2x-1)(x+2)}$$

$$\frac{(2x+3)(2x-1) - (9+2x)}{(x+2)(2x-1)}$$

$$\frac{(2x+3)(2x-1) - (9+2x)}{(x+2)(2x-1)}$$

$$\frac{4x^2 - 2x + 6x - 3 - 9 - 2x}{(x+2)(2x-1)}$$

$$\frac{4x^2 + 2x - 12}{(x+2)(2x-1)}$$

$$\frac{2(2x-3)(x+2)}{(x+2)(2x-1)}$$

$$\frac{2(2x-3)}{(2x-1)}$$

$$\frac{4x-6}{2x-1}$$

b)

$$u = 4x - 6$$

$$v = 2x - 1$$

$$\frac{du}{dx} = 4$$

$$\frac{dv}{dx} = 2$$

$$\frac{dy}{dx} = \frac{4(2x-1) - 2(4x-6)}{(2x-1)^2}$$

$$= \frac{8x - 4 - 8x + 12}{(2x-1)^2}$$

$$= \frac{8}{(2x-1)^2}$$