Name: ______________________________

GCSE (1 – 9)

Iteration

Instructions

• Use black ink or ball-point pen.
• Answer all questions.
• Answer the questions in the spaces provided
  – there may be more space than you need.
• Diagrams are NOT accurately drawn, unless otherwise indicated.
• You must show all your working out.

Information

• The marks for each question are shown in brackets
  – use this as a guide as to how much time to spend on each question.

Advice

• Read each question carefully before you start to answer it.
• Keep an eye on the time.
• Try to answer every question.
• Check your answers if you have time at the end

maths genie.co.uk
1 The number of rabbits in a field $t$ days from now is $P_t$, where

\[
P_0 = 220 \\
P_{t+1} = 1.15(P_t - 20)
\]

Work out the number of rabbits in the garden 3 days from now.

\[
P_1 = 1.15 \left( 220 - 20 \right) = 230 \\
P_2 = 1.15 \left( \text{Ans} - 20 \right) = 242 \\
P_3 = 1.15 \left( \text{Ans} - 20 \right) = 255 \quad \text{(nearest integer)}
\]

(Total for question 1 is 3 marks)

2 The number of people living in a town $t$ years from now is $P_t$, where

\[
P_0 = 55000 \\
P_{t+1} = 1.03(P_t - 800)
\]

Work out the number of people in the town 3 years from now.

\[
P_1 = 1.03 \left( 55000 - 800 \right) = 55826 \\
P_2 = 1.03 \left( \text{Ans} - 800 \right) = 56677 \\
P_3 = 1.03 \left( \text{Ans} - 800 \right) = 57553 \quad \text{(nearest integer)}
\]

(Total for question 2 is 3 marks)
3 Using \( x_{n+1} = 3 + \frac{9}{x_n^2} \)

With \( x_0 = 3 \)

Find the values of \( x_1, x_2 \) and \( x_3 \).

\[
\begin{align*}
    x_1 &= 3 + \frac{9}{(3)^2} = 4 \\
    x_2 &= 3 + \frac{9}{(\text{ans})^2} = 3.5625 \\
    x_3 &= 3 + \frac{9}{(\text{ans})^2} = 3.709141274
\end{align*}
\]

\( x_1 = \) \underline{4} \\
\( x_2 = \) \underline{3.5625} \\
\( x_3 = \) \underline{3.709141274}

(Total for question 3 is 3 marks)

4 Using \( x_{n+1} = -\frac{5}{x_n^2 + 3} \)

With \( x_0 = 1 \)

Find the values of \( x_1, x_2 \) and \( x_3 \).

\[
\begin{align*}
    x_1 &= \frac{5}{(1)^2 + 3} = 1.25 \\
    x_2 &= \frac{5}{(\text{ans})^2 + 3} = 1.095890411 \\
    x_3 &= \frac{5}{(\text{ans})^2 + 3} = 1.190199669
\end{align*}
\]

\( x_1 = \) \underline{1.25} \\
\( x_2 = \) \underline{1.095890411} \\
\( x_3 = \) \underline{1.190199669}

(Total for question 4 is 3 marks)
5 Starting with \( x_0 = 3 \), use the iteration formula \( x_{n+1} = \frac{7}{x_n^2} + 2 \) three times to find an estimate for the solution to \( x^3 - 2x^2 = 7 \)

\[
x_1 = \frac{7}{(3)^2} + 2 = \frac{25}{9}
\]
\[
x_2 = \frac{7}{(\text{Ans})^2} + 2 = 2.9072
\]
\[
x_3 = \frac{7}{(\text{Ans})^2} + 2 = 2.82822478
\]

(Total for question 5 is 3 marks)

6 Starting with \( x_0 = 0 \), use the iteration formula \( x_{n+1} = \frac{2}{x_n^2 + 3} \) three times to find an estimate for the solution to \( x^3 + 3x = 2 \)

\[
x_1 = \frac{2}{(0)^2 + 3} = \frac{2}{3}
\]
\[
x_2 = \frac{2}{(\text{Ans})^2 + 3} = \frac{18}{31}
\]
\[
x_3 = \frac{2}{(\text{Ans})^2 + 3} = 0.5993140006
\]

(Total for question 6 is 3 marks)
Using \( x_{n+1} = \frac{5}{x_n^2} + 2 \)

With \( x_0 = 2.5 \)

(a) Find the values of \( x_1 \), \( x_2 \) and \( x_3 \).

\[
x_1 = \frac{5}{(2.5)^2} + 2 = 2.8
\]

\[
x_2 = \frac{5}{(2.637755102)^2} + 2 = 2.718622914
\]

\[
x_3 = \frac{5}{(2.718622914)^2} + 2
\]

\[
x_1 = 2.8
\]

\[
x_2 = 2.637755102
\]

\[
x_3 = 2.718622914
\]

(b) Explain the relationship between the values of \( x_1 \), \( x_2 \) and \( x_3 \) and the equation \( x^3 - 2x^2 - 5 = 0 \)

\[
x^2(x - 2) - 5 = 0
\]

\[
x^2(x - 2) = 5
\]

\[
x - 2 = \frac{5}{x^2}
\]

\[
x = \frac{5}{x^2} + 2
\]

\( x = \frac{5}{x^2} + 2 \) is a rearrangement of \( x^3 - 2x^2 - 5 = 0 \)

\( x_1 \), \( x_2 \) and \( x_3 \) are estimates of the solution to \( x^3 - 2x^2 - 5 = 0 \)

(Total for question 7 is 5 marks)
8 (a) Show that the equation $2x^3 - x^2 - 3 = 0$ has a solution between $x = 1$ and $x = 2$.

\[ \text{when } x = 1 \quad 2(1)^3 - (1)^2 - 3 = -2 \]
\[ \text{when } x = 2 \quad 2(2)^3 - (2)^2 - 3 = 9 \]

one positive and one negative: solution between 1 and 2. \( (2) \)

(b) Show that the equation $2x^3 - x^2 - 3 = 0$ can be rearranged to give: $x = \sqrt[3]{3} \div (2x - 1)$

\[ x^2 (2x - 1) - 3 = 0 \]
\[ x^2 (2x - 1) = 3 \]
\[ x^2 = \frac{3}{2x - 1} \]
\[ x = \sqrt[3]{\frac{3}{2x - 1}} \] \( (1) \)

(c) Starting with $x_0 = 1$, use the iteration formula $x_{n+1} = \sqrt[3]{\frac{3}{2x_n - 1}}$ twice to find an estimate for the solution to $2x^3 - x^2 - 3 = 0$

\[ x_1 = \sqrt[3]{\frac{3}{2(1) - 1}} = \sqrt[3]{3} \]
\[ x_2 = \sqrt[3]{\frac{3}{2(Ans) - 1}} = 1.103395785 \]

\[ \text{1.103395785} \]

\( (3) \)

(Total for question 8 is 6 marks)
Using \( x_{n+1} = 1 + \frac{1}{x_n^2} \)

With \( x_0 = 2 \)

(a) Find the values of \( x_1, x_2 \) and \( x_3 \).

\[
\begin{align*}
  x_1 &= 1 + \frac{1}{(2)^2} = 1.25 \\
  x_2 &= 1 + \frac{1}{(1.25)^2} = 1.64 \\
  x_3 &= 1 + \frac{1}{(1.64)^2} = 1.871802499
\end{align*}
\]

\[
\begin{align*}
  x_1 &= 1.25 \\
  x_2 &= 1.64 \\
  x_3 &= 1.871802499
\end{align*}
\]

(b) Explain the relationship between the values of \( x_1, x_2 \) and \( x_3 \) and the equation \( x^3 - x^2 - 1 = 0 \)

\[
\begin{align*}
  x^2(x - 1) - 1 &= 0 \\
  x^2(x - 1) &= 1 \\
  x - 1 &= \frac{1}{x^2} \\
  x &= 1 + \frac{1}{x^2}
\end{align*}
\]

\( x = 1 + \frac{1}{x^2} \) is a rearrangement of \( x^3 - x^2 - 1 = 0 \)

\( x_1, x_2, \) and \( x_3 \) are estimates of a solution to \( x^3 - x^2 - 1 = 0 \).

(Total for question 9 is 5 marks)
10 (a) Show that the equation \( x^3 + 4x = 1 \) has a solution between \( x = 0 \) and \( x = 1 \).

\[
x^3 + 4x - 1 = 0
\]

when \( x = 0 \) \( (0)^3 + 4(0) - 1 \) = \(-1\)

\( x = 1 \) \( (1)^3 + 4(1) - 1 \) = \(4\)

one positive, one negative \( \therefore \) solution between \( 0 \) and \( 1 \)

(b) Show that the equation \( x^3 + 4x = 1 \) can be rearranged to give: \( x = \frac{1}{4} \left( \frac{-x^3}{4} \right) \)

\[
4x = 1 - x^3
\]

\[
x = \frac{1}{4} - \frac{x^3}{4}
\]

(c) Starting with \( x_0 = 0 \), use the iteration formula \( x_{n+1} = \frac{1}{4} - \frac{x_n^3}{4} \) twice to find an estimate for the solution to \( x^3 + 4x = 1 \)

\[
x_1 = \frac{1}{4} - \frac{(0)^3}{4} = 0.25
\]

\[
x_2 = \frac{1}{4} - \frac{(Ans)^3}{4} = 0.24609375
\]

\(0.24609375\) (Total for question 10 is 6 marks)