1 $y = 2x^3 + 5x^2 - 7x + 10$

(a) Find $\dfrac{dy}{dx}$ **(2)**

(b) Find the gradient of the curve when $x = 2$ **(1)**

a) $\dfrac{dy}{dx} = 6x^2 + 10x - 7$

b) when $x = 2$ $\dfrac{dy}{dx} = 6(2)^2 + 10(2) - 7$

$$= 37$$

---

2 $y = 3x + \dfrac{1}{x}$

(a) Find $\dfrac{dy}{dx}$ **(2)**

(b) Find the x coordinates of points where the gradient is zero.   **(2)**

a) $y = 3x + x^{-1}$

$\dfrac{dy}{dx} = 3 - x^{-2}$

b) $3 - x^{-2} = 0$

$$3 = x^{-2}$$

$$3 = \dfrac{1}{x^2}$$

$$x^2 = \dfrac{1}{3}$$

$$x = \pm\sqrt{\dfrac{1}{3}}$$

---

3 $f(x) = 3x^{\frac{3}{2}} + \dfrac{3}{x^2} - 6x$

Find $f'(x)$

$f(x) = 3x^{\frac{3}{2}} + 3x^{-2} - 6x$

$f'(x) = \dfrac{9}{2}x^{\frac{1}{2}} - 6x^{-3} - 6$

4 $\quad y = 4\sqrt{x} + \dfrac{1}{2x} + 10$

(a) Find $\dfrac{dy}{dx}$ (3)

(b) Find $\dfrac{d^2y}{dx^2}$ (2)

$$y = 4x^{\frac{1}{2}} + \frac{1}{2}x^{-1} + 10$$

a/ $\quad \dfrac{dy}{dx} = 2x^{-\frac{1}{2}} - \dfrac{1}{2}x^{-2}$

b/ $\quad \dfrac{d^2y}{dx^2} = -x^{-\frac{3}{2}} + x^{-3}$

---

5 $\quad y = \dfrac{2x^2 - 5x + 3}{x}$

(a) Find $\dfrac{dy}{dx}$ (3)

(b) Find the gradient when $x = 3$ (1)

$$y = 2x - 5 + 3x^{-1}$$

a/ $\quad \dfrac{dy}{dx} = 2 - 3x^{-2}$

b/ when $x = 3 \qquad \dfrac{dy}{dx} = 2 - 3(3)^{-2}$

$$= \frac{5}{3}$$

---

6 $\quad y = x^3 - 4x^2 - 3x + 9$

(a) Find $\dfrac{dy}{dx}$ (2)

(b) Find the range values of x for which y is increasing   (3)

a/ $\quad \dfrac{dy}{dx} = 3x^2 - 8x - 3$

b/ $\quad 3x^2 - 8x - 3 > 0$

$$x < -\frac{1}{3} \quad \text{or} \quad x > 3$$

7 A curve has the equation $y = 2x^3 + 9x^2 - 24x + 13$

Find the coordinates of the curve's local maximum.

$$\frac{dy}{dx} = 6x^2 + 18x - 24$$

turning point where $\frac{dy}{dx} = 0$

$$6x^2 + 18x - 24 = 0$$
$$x^2 + 3x - 4 = 0$$
$$(x + 4)(x - 1) = 0$$
$$x = -4 \qquad x = 1$$

$$\frac{d^2y}{dx^2} = 12x + 18$$

when $x = -4$ $\quad \frac{d^2y}{dx^2} = -30$ (maximum)

$x = 1$ $\quad \frac{d^2y}{dx^2} = 30$ (minimum)

$$y = 2(-4)^3 + 9(-4)^2 - 24(-4) + 13$$
$$= 125$$

$$\underline{\underline{(-4, 125)}}$$

8
$$y = 4x^2 + \frac{16}{x} + 1 \qquad x > 0$$

(a) Find $\frac{dy}{dx}$ **(3)**

(b) Find the exact range of values of x for which the curve is increasing. **(2)**

a/ $\qquad y = 4x^2 + 16x^{-1} + 1$

$$\frac{dy}{dx} = 8x - 16x^{-2}$$

b/ $\qquad 8x - 16x^{-2} > 0$

$$8x - \frac{16}{x^2} > 0$$

$$8x^3 - 16 > 0$$
$$x^3 > 2$$
$$x > \sqrt[3]{2}$$

9   A curve has the equation $y = 2x^3 - 12x^2 + 18x + 5$

    (a) The curve has a local minimum at P, find the coordinates of P.   (4)

    (b) Justify that P is a minimum point.   (2)

a/ $\dfrac{dy}{dx} = 6x^2 - 24x + 18$

Minimum where $\dfrac{dy}{dx} = 0$

$$6x^2 - 24x + 18 = 0$$
$$x^2 - 4x + 3 = 0$$
$$(x - 3)(x - 1) = 0$$
$$x = 3 \qquad x = 1$$

$\dfrac{d^2y}{dx^2} = 12x - 24$

when $x = 3$ $\dfrac{d^2y}{dx^2} = 12$ positive $\therefore$ minimum.

$y = 2(3)^3 - 12(3)^2 + 18(3) + 5$
$ = 5$
$$\underline{\underline{(3, 5)}}$$

b/ $\dfrac{d^2y}{dx^2} = 12x - 24$

when $x = 3$ $\dfrac{d^2y}{dx^2} = 12$ positive $\therefore$ minimum.

10   A curve has the equation $y = 3x^2 - 5x + 7$

Find the equation of the tangent to the curve at the point $P(2, 9)$.

Write your answer in the form $y = mx + c$, where m and c are integers to be found.

$$\frac{dy}{dx} = 6x - 5$$

when $x = 2$ $\frac{dy}{dx} = 6(2) - 5$

$$= 7 \qquad\qquad m = 7 \qquad\qquad \underset{x \quad y}{(2, 9)}$$

$$y - 9 = 7(x - 2)$$
$$y - 9 = 7x - 14$$
$$y = 7x - 5$$

---

11   A curve has the equation $y = g(x)$

Given that
- $g(x)$ is a cubic expression in which the coefficient of x^3 is equal to the coefficient of x
- the curve with equation $y = g(x)$ passes through the origin
- the curve with equation $y = g(x)$ has a stationary point at $(2, -10)$

(a) Find $g(x)$ (7)

(b) prove that the stationary point at $(2, -10)$ is a minimum. $c = 0$ (2)

$$g(x) = ax^3 + bx^2 + ax + c$$
$$g(x) = ax^3 + bx^2 + ax$$
$$g'(x) = 3ax^2 + 2bx + a$$

stationary point when $x = 2$ $3a(2)^2 + 2b(2) + a = 0$

$$12a + 4b + a = 0$$
$$13a + 4b = 0 \;\;②$$

$$-10 = a(2)^3 + b(2)^2 + a(2)$$
$$-10 = 8a + 4b + 2a$$
$$10a + 4b = -10 \;\;①$$

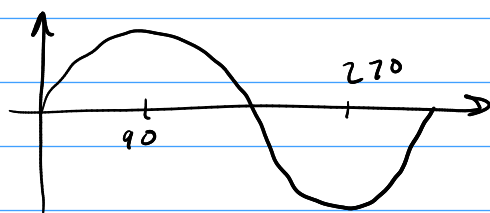$$a = \frac{10}{3} \qquad b = \frac{-65}{6}$$

$$g(x) = \frac{10}{3}x^3 - \frac{65}{6}x^2 + \frac{10}{3}$$

b/ $g''(x) = 6ax + 2b$

$$= 20x - \frac{65}{3}$$

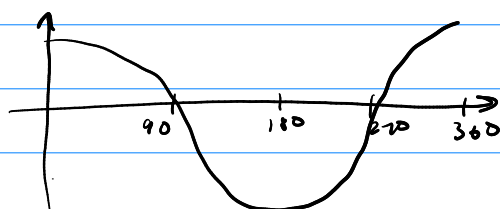$$g''(2) = 20(2) - \frac{65}{3} = \frac{55}{3}$$

positive $\therefore$ minimum

12 State the interval for which $y = \sin x$ is a decreasing function for $0° \leq x \leq 360°$

decreasing when the gradient is negative

$$90 < x < 270$$

13 State the interval for which $y = \cos x$ is an increasing function for $0° \leq x \leq 360°$

$$180 < x < 360$$

14 $y = 2x^3 + 5x^2 - 7x + 10$

Find the equation of the tangent at the point where $x = 1$
Give your answer in the form $y = mx + c$

$$\frac{dy}{dx} = 6x^2 + 10x - 7$$

when $x = 1$ $\quad \frac{dy}{dx} = 6(1)^2 + 10(1) - 7$
$$= 9 \qquad\qquad m = 9$$

$$y = 2(1)^3 + 5(1)^2 - 7(1) + 10$$
$$= 10 \qquad\qquad (1, 10)$$

$$y - 10 = 9(x - 1)$$
$$= 9x - 9$$
$$y = 9x + 1$$

15 $f(x) = 2x^3 + x^2 - 18x + 2$

The points A and B lie on the curve $y = f(x)$. The gradient at both A and B is 2.
Find the coordinates of A and B.

$$f'(x) = 6x^2 + 2x - 18$$
$$2 = 6x^2 + 2x - 18$$
$$0 = 6x^2 + 2x - 20$$
$$0 = 3x^2 + x - 10$$
$$0 = (3x - 5)(x + 2)$$
$$x = \frac{5}{3} \qquad x = -2$$

$$y = 2\left(\frac{5}{3}\right)^3 + \left(\frac{5}{3}\right)^2 - 18\left(\frac{5}{3}\right) + 2 \qquad \left(\frac{5}{3}, \frac{-431}{27}\right)$$

$$y = 2(-2)^3 + (-2)^2 - 18(-2) + 2 \qquad (-2, 26)$$

---

16 $y = \dfrac{(4x - 1)(x + 2)}{2x}$

Find the equation of the normal at the point when $x = -2$
Give your answer in the form $ax + by + c = 0$ where a, b and c are integers.

$$y = \frac{4x^2 + 8x - x - 2}{2x}$$

when $x = -2$

$$y = \frac{(4(-2) - 1)(-2 + 2)}{2(-2)}$$

$$= \frac{4x^2 + 7x - 2}{2x}$$

$$= 0$$

$$= 2x + \frac{7}{2} - x^{-1}$$

$$(-2, 0)$$
$$x_1 \quad y_1$$

$$\frac{dy}{dx} = 2 + x^{-2}$$

when $x = -2$ $\qquad \dfrac{dy}{dx} = 2 + (-2)^{-2}$

$$= \frac{9}{4}$$

perp $M = -\frac{4}{9}$

$$y - 0 = -\frac{4}{9}(x + 2)$$

$$9y = -4x - 8$$
$$4x + 9y + 8 = 0$$

17 A simple model for the cost of a car journey £C when a car is driven at a steady speed of v mph is

$$C = \frac{4500}{v} + v + 10$$

(a) Use this model to find the value of v which minimises the cost of the journey. **(5)**

(b) Use $\frac{d^2C}{dv^2}$ to verify that C is a minimum for this value of v **(2)**

(c) Calculate the minimum cost of the journey **(2)**

a) $\quad C = 4500v^{-1} + v + 10$

$\frac{dC}{dv} = -4500v^{-2} + 1$

$-4500v^{-2} + 1 = 0$

$1 = \frac{4500}{v^2}$

$v^2 = 4500$

$v = 67.1 \text{ mph}$

b) $\quad \frac{d^2C}{dv^2} = 9000v^{-3}$

when $v = 67.1$ $\qquad \frac{d^2C}{dv^2} = 0.0298$ $\qquad$ +ve ∴ Minimum

c) $\quad C = \frac{4500}{67.1} + 67.1 + 10$

$= £144 \ (3sf)$

18 A cylinder has a radius r and a height h.
The surface area of the cylinder is 500cm²

(a) Show that the volume (V cm³) of the cylinder is given by $V = 250r - \pi r^3$ **(4)**

Given that r varies

(b) Calculate the maximum value of V, to the nearest cm³ **(6)**

(c) Justify that the value of V you found is a maximum. **(2)**

a/

$$volume = \pi r^2 h \qquad\qquad s.a = 2\pi r^2 + 2\pi rh$$

$$500 = 2\pi r^2 + 2\pi rh$$

$$250 = \pi r^2 + \pi rh$$

$$250 - \pi r^2 = \pi rh$$

$$h = \frac{250 - \pi r^2}{\pi r}$$

$$volume = \pi r^2 \left(\frac{250 - \pi r^2}{\pi r} \right)$$

$$= \frac{250 \pi r^2 - \pi^2 r^4}{\pi r}$$

$$= \underline{250r - \pi r^3}$$

b/

$$\frac{dV}{dr} = 250 - 3\pi r^2$$

max when

$$250 - 3\pi r^2 = 0$$

$$250 = 3\pi r^2$$

$$\frac{250}{3\pi} = r^2$$

$$r = \sqrt{\frac{250}{3\pi}}$$

$$= \underline{\underline{5.15}}$$

$$v = 250(5.15) - \pi(5.15)^3$$

$$= \underline{\underline{858 \ cm^3}}$$

c/

$$\frac{d^2v}{dr^2} = -6\pi r \qquad when \ r = 5.15$$

$$\frac{d^2v}{dr^2} = -97.1 \qquad \underline{negative} \ \therefore \ \underline{maximum}$$

19   A curve has the equation $y = 4x^3 + 15x^2 - 18x + 5$

Find the coordinates of the stationary points and determine the nature of each stationary point.

$$\frac{dy}{dx} = 12x^2 + 30x - 18$$

stationary points where $\frac{dy}{dx} = 0$

$$12x^2 + 30x - 18 = 0$$

$$x = \frac{1}{2} \qquad x = -3$$

$$y = \frac{1}{4} \qquad y = 86$$

$$\left(\frac{1}{2}, \frac{1}{4}\right) \qquad (-3, 86)$$

$$\frac{d^2y}{dx^2} = 24x + 30$$

when $x = \frac{1}{2}$ $\frac{d^2y}{dx^2} = 42$ positive $\therefore$ Minimum at

$$\left(\frac{1}{2}, \frac{1}{4}\right)$$

when $x = -3$ $\frac{d^2y}{dx^2} = -42$ negative $\therefore$ Maximum at $(-3, 86)$

20 A curve has equation $y = 3x^2 - 16x\sqrt{x} + 18x - 2$ for $x \geq 0$

(a) Prove that the curve has a maximum point at $(1, 3)$ **(9)**
Fully justify your answer.

(b) Find the coordinates of the other stationary point of the curve and state its nature. **(2)**

$$y = 3x^2 - 16x^{\frac{3}{2}} + 18x - 2$$

$$\frac{dy}{dx} = 6x - 24x^{\frac{1}{2}} + 18$$

Maximum point when $\frac{dy}{dx} = 0$

$$6x - 24x^{\frac{1}{2}} + 18 = 0$$
$$x - 4x^{\frac{1}{2}} + 3 = 0$$
$$\left(x^{\frac{1}{2}} - 3\right)\left(x^{\frac{1}{2}} - 1\right) = 0$$

$$x^{\frac{1}{2}} = 3 \qquad x^{\frac{1}{2}} = 1$$
$$\underline{x = 9} \qquad \underline{x = 1}$$

$$y = 3(1)^2 - 16(1)^{\frac{3}{2}} + 18(1) - 2$$
$$\underline{= 3}$$

$$\underline{\underline{(1, 3)}}$$

$$\frac{d^2y}{dx^2} = 6 - 12x^{-\frac{1}{2}}$$

when $x = 1$ $\qquad \frac{d^2y}{dx^2} = 6 - 12(1)^{-\frac{1}{2}}$

$$= -6$$

$$\underline{negative \therefore minimum}$$

b/ $\quad y = 3(9)^2 - 16(9)^{\frac{3}{2}} + 18(9) - 2$
$\quad\quad = -29$

$\underline{(9, -29)} \qquad \frac{d^2y}{dx^2} = 6 - 12(9)^{-\frac{1}{2}} = 2 \qquad$ Positive $\therefore$ MINIMUM

21 A company is designing a cup. The cup will be in the shape of a cylinder with radius x and height h.

The cup does not have a lid and must hold 450 ml of liquid.

(a) Show that the surface area of the cup is given by $\pi x^2 + \dfrac{900}{x}$ (4)

(b) Find, to 2 decimal places, the value of x that makes the surface area a minimum. (4)

(c) Justify that the value of x you found is a minimum. (2)

(d) Give a reason why the company may not choose to make a cup with a radius this size. (1)

$$\text{Volume} = \pi r^2 h \qquad\qquad \text{S.a} = \pi r^2 + 2\pi rh$$

$$450 = \pi r^2 h \qquad\qquad\qquad = \pi r^2 + 2\pi r \cdot \dfrac{450}{\pi r^2}$$

$$\dfrac{450}{\pi r^2} = h$$

$$= \pi r^2 + \dfrac{900}{r}$$

$$= \pi x^2 + \dfrac{900}{x} \qquad (r = x)$$

b/ $$\dfrac{dS}{dx} = 2\pi x - 900 x^{-2}$$

min where $\dfrac{dS}{dx} = 0 \qquad 2\pi x - \dfrac{900}{x^2} = 0$

$$2\pi x = \dfrac{900}{x^2}$$

$$2\pi x^3 = 900$$

$$x^3 = \dfrac{900}{2\pi}$$

$$x = 5.23 \ (cm)$$

c/ $$\dfrac{d^2S}{dx^2} = 2\pi + 1800 x^{-3}$$

when $x = 5.23 \qquad \dfrac{d^2S}{dx^2} = 6\pi$ \qquad positive ∴ minimum

d/ It would be too wide. It may spill easily or be difficult to hold.

22 Prove that the curve with equation

$$y = 4x^5 + 15x^4 + 20x^3 + 7$$

only has one stationary point, stating its coordinates.

$$\frac{dy}{dx} = 20x^4 + 60x^3 + 60x^2$$

stationary point where $\frac{dy}{dx} = 0$

$$20x^4 + 60x^3 + 60x^2 = 0$$
$$x^4 + 3x^3 + 3x^2 = 0$$
$$x^2(x^2 + 3x + 3) = 0$$
$$x = 0 \quad \text{or} \quad x^2 + 3x + 3 = 0$$

$$(3)^2 - 4(1)(3) = -3$$

$b^2 - 4ac < 0$

$$\therefore \text{ no solutions}$$

when $x = 0$ $\quad y = 4(0)^5 + 15(0)^4 + 20(0)^3 + 7$
$$= 7$$

$$\underline{\underline{(0, 7)}}$$

23 A curve has equation

$$y = 2x^3 - 3x^2 + 4x - 5$$

(a) Find $\dfrac{dy}{dx}$ **(4)**

(b) Show that the perpendicular bisector of the line joining A(6, 2) and B(4, -6) is a normal to the curve at (1, -1).   **(6)**

a/ $\dfrac{dy}{dx} = 6x^2 - 6x + 4$

b/   when $x = 1$ $\dfrac{dy}{dx} = 4$ $\underset{\underset{x_1 \quad y_1}{}}{(1, -1)}$

$$\therefore m = -\frac{1}{4}$$

$$y + 1 = -\frac{1}{4}(x - 1)$$

$$4y + 4 = -(x - 1)$$

$$4y + 4 = -x + 1$$

$$\underline{\underline{x + 4y + 3 = 0}}$$

Midpoint of $AB = \left(\dfrac{6+4}{2}, \dfrac{2-6}{2} \right)$

$$= \underset{\underset{x_1 \quad y_1}{}}{(5, -2)}$$

$$m_{AB} = \frac{-6 - 2}{4 - 6} = \frac{-8}{-2} = 4$$

perp. $m = -\dfrac{1}{4}$

$$y + 2 = -\frac{1}{4}(x - 5)$$

$$4y + 8 = -(x - 5)$$

$$4y + 8 = -x + 5$$

$$\underline{\underline{x + 4y + 3 = 0}}$$

24 A curve has equation

$$y = x^3 + px^2 + qx - 5$$

The curve passes through the point A (2, 1)
$\quad x \quad y$

The gradient of the curve at A is 5.

Find the value of p and the value of q.

$$1 = (2)^3 + p(2)^2 + q(2) - 5$$
$$1 = 8 + 4p + 2q - 5$$
$$-2 = 4p + 2q$$
$$-1 = 2p + q \qquad ①$$

$$\frac{dy}{dx} = 3x^2 + 2px + q \qquad \text{when } x = 2 \quad \frac{dy}{dx} = 5$$

$$5 = 3(2)^2 + 2p(2) + q$$
$$5 = 12 + 4p + q$$
$$-7 = 4p + q \qquad ②$$

$$p = -3 \qquad q = 5$$

25 A curve has equation $f(x) = (x + 3)(x - 2)^2$

(a) Find the coordinates of the turning points of the curve.
Determine the nature of each turning point. **(8)**

(b) State the coordinates of the turning points of the curve $y = 2f(x - 1)$ **(2)**

a/
$$f(x) = (x + 3)(x^2 - 4x + 4)$$
$$= x^3 + 3x^2 - 4x^2 - 12x + 4x + 12$$
$$= x^3 - x^2 - 8x + 12$$

$$f'(x) = 3x^2 - 2x - 8$$

turning points where $f'(x) = 0$

$$3x^2 - 2x - 8 = 0$$
$$(3x + 4)(x - 2) = 0$$

$$x = -\frac{4}{3} \qquad x = 2$$

$$y = \frac{500}{27} \qquad y = 0$$

$$\left(-\frac{4}{3}, \frac{500}{27}\right) \qquad (2, 0)$$

$$f''(x) = 6x - 2$$

when $x = -\frac{4}{3}$ $\qquad f''(x) = -10$ $\qquad f''(x) < 0$
$$\therefore \text{ Local Maximum}$$

when $x = 2$ $\qquad f''(x) = 10$ $\qquad f''(x) > 0$
$$\therefore \text{ Local Minimum.}$$

26 A curve has equation $y = 3x^4 - 2\sqrt{x} + \frac{x}{2} - 2$

Find an expression for $\frac{d^2y}{dx^2}$

$$y = 3x^4 - 2x^{\frac{1}{2}} + \frac{1}{2}x - 2$$

$$\frac{dy}{dx} = 12x^3 - x^{-\frac{1}{2}} + \frac{1}{2}$$

$$\frac{d^2y}{dx^2} = 36x^2 + \frac{1}{2}x^{-\frac{3}{2}}$$

---

27 A curve has equation $\qquad y = ax^2 - \frac{b}{\sqrt{x}} + \frac{c}{x}$

(a) In terms of a, b and c, find an expression for $\frac{dy}{dx}$ **(4)**

(b) In terms of a, b and c, find an expression for $\frac{d^2y}{dx^2}$ **(3)**

$$y = ax^2 - bx^{-\frac{1}{2}} + cx^{-1}$$

a/

$$\frac{dy}{dx} = 2ax + \frac{1}{2}x^{-\frac{3}{2}} - cx^{-2}$$

b/

$$\frac{d^2y}{dx^2} = 2a - \frac{3}{4}x^{-\frac{5}{2}} + 2cx^{-3}$$

28 A curve has equation $y = x^2 - 4x$

(a) Find $\dfrac{dy}{dx}$ (2)

(b) Find the values of x for which y is increasing.   (2)

a/ $\dfrac{dy}{dx} = 2x - 4$

b/ $2x - 4 > 0$

$2x > 4$

$\underline{\underline{x > 2}}$

---

29 The line $y = 3x + k$ is a tangent to the curve $x^2 - y = 3$. Find the value of the constant k.

one solution to sim. eq.

$$x^2 - (3x + k) = 3$$

$$x^2 - 3x - k = 3$$
$$x^2 - 3x - k - 3 = 0$$

$$b^2 - 4ac = 0$$

$$(-3)^2 - 4(1)(-k-3) = 0$$
$$9 - 4(-k - 3) = 0$$
$$9 + 4k + 12 = 0$$
$$4k + 21 = 0$$
$$\underline{\underline{k = -\dfrac{21}{4}}}$$

30 Find the equation of the normal to the curve $y = 2\sqrt{x} + 3x + 1$ at the point where $x = 4$.

Give your answer in the form $ax + by + c = 0$, where a, b and c are integers.

$$y = 2x^{\frac{1}{2}} + 3x + 1$$

$$\frac{dy}{dx} = x^{-\frac{1}{2}} + 3$$

when $x = 4$ $\qquad \frac{dy}{dx} = (4)^{-\frac{1}{2}} + 3$

$$= \frac{7}{2}$$

gradient of normal $= -\frac{2}{7} \, m$

when $x = 4$ $\qquad y = 2\sqrt{4} + 3(4) + 1$

$$x \qquad\qquad = 17$$
$$y$$

$$y - 17 = -\frac{2}{7}(x - 4)$$

$$7(y - 17) = -2(x - 4)$$

$$7y - 119 = -2x + 8$$

$$\underline{\underline{2x + 7y - 127 = 0}}$$

31 Find the equation of the normal to the curve $y = (2x - 1)^2$ at the point where $x = 2$.

Give your answer in the form $y = mx + c$

$$y = 4x^2 - 4x + 1$$

$$\frac{dy}{dx} = 8x - 4$$

when $x = 2 \qquad \frac{dy}{dx} = 12$

$$\therefore \quad m = -\frac{1}{12}$$

when $x = 2 \qquad y = (2(2) - 1)^2$
$$= 9$$

$$y - 9 = -\frac{1}{12}(x - 2)$$

$$y - 9 = -\frac{1}{12}x + \frac{1}{6}$$

$$y = -\frac{1}{12}x + \frac{55}{6}$$

32 (a) Sketch the gradient function of the curve $y = x^3 - 3x^2 - 45x$ (5)

(b) Determine the set of values for which $x^3 - 3x^2 - 45x$ is decreasing (2)

a) $\dfrac{dy}{dx} = 3x^2 - 6x - 45$

sketch $y = 3x^2 - 6x - 45$
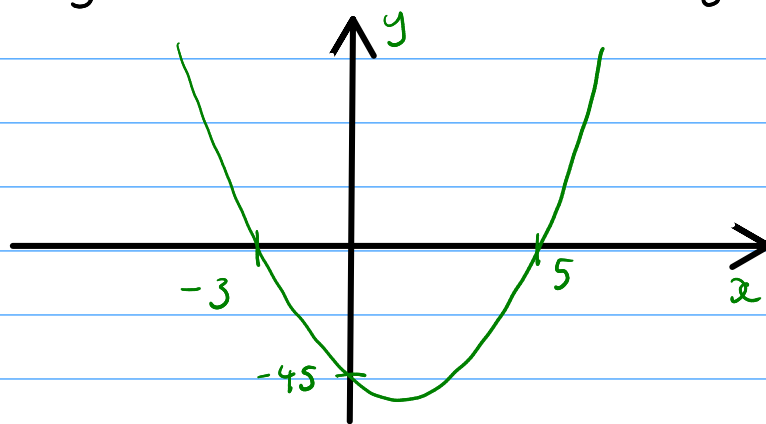
crosses x when $y = 0$

$$0 = 3x^2 - 6x - 45$$
$$0 = x^2 - 2x - 15$$
$$0 = (x - 5)(x + 3)$$
$$x = 5 \qquad x = -3$$

crosses y when $x = 0$ $\qquad\qquad y = -45$

b) $x^3 - 3x^2 - 45x$ is decreasing where the gradient function is below the x-axis.

$$-3 < x < 5$$

33 The equation of a curve is $y = 2x^2 + \dfrac{1}{x}$

A tangent and a normal to the curve are drawn at the point where $x = 1$.

Calculate the area bounded by the tangent, the normal and the x-axis.

when $x = 1$ $\quad y = 2(1)^2 + \dfrac{1}{1}$

$\qquad\qquad\qquad = 3$

$$y = 2x^2 + x^{-1}$$

$$\frac{dy}{dx} = 4x - x^{-2}$$

when $x = 1$ $\quad \dfrac{dy}{dx} = 5$

gradient of tangent $= 5$

gradient of normal $= -\dfrac{1}{5}$

Tangent

$y - 3 = 5(x - 1)$
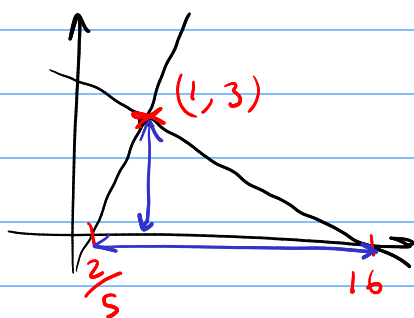
$y - 3 = 5x - 5$

$y = 5x - 2$

Normal

$y - 3 = -\dfrac{1}{5}(x - 1)$

$y - 3 = -\dfrac{1}{5}x + \dfrac{1}{5}$

$y = -\dfrac{1}{5}x + \dfrac{16}{5}$

lines cross x when $y = 0$

$0 = 5x - 2$

$2 = 5x$

$x = 2/5$

$0 = -\dfrac{1}{5}x + \dfrac{16}{5}$

$0 = -x + 16$

$x = 16$

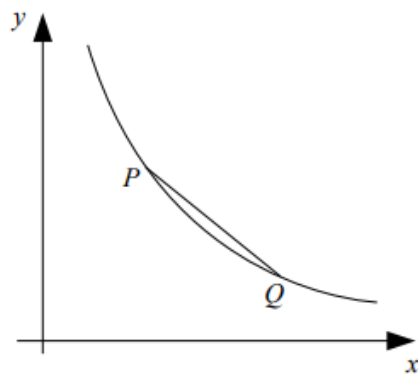Base $= 16 - \dfrac{2}{5}$

$\quad = \dfrac{78}{5}$

Height $= 3$

Area $= \dfrac{1}{2} \cdot \dfrac{78}{5} \cdot 3 = \dfrac{117}{5}$ units^2

34   The graph shows part of the curve with equation $y = \dfrac{4}{x}$

P is the point with coordinates (1, 4) and Q is the point with x coordinate (1+ h)

The table shows for different values of h, the coordinates of P, the coordinates of Q and the gradient of the chord PQ

x for P	y for P	h	x for Q	y for Q	Gradient
1	4	1	2	2	-2
1	4	0.1	1.1	3.636364	-3.636364
1	4	0.01	1.01	3.960396	-3.960396
1	4	0.001	1.001	3.996004	-3.996004

(a) Complete the table.   (3)

(b) Explain how the sequence of values in the last column relates to the gradient of the
curve at the point P.   (1)

(c) Use calculus to find the gradient of the curve at the point P.   (2)

b/ The gradient is getting closer and closer to the gradient at P.

c/ $y = 4x^{-1}$

        $\dfrac{dy}{dx} = -4x^{-2}$

when $x = 1$ $\dfrac{dy}{dx} = -4(1)^{-2}$

                                    $= -4$

35 Danny is investigating the gradient of chords of the curve with equation $f(x) = 2 - x^2$

Each chord joins the point (3, -7) to the point (3 + h, f(3 + h))

The table shows Danny's results.

x	f(x)	h	x + h	f(x + h)	Gradient
3	-7	1	4	-14	-7
3	-7	0.1	3.1	-7.61	-6.1
3	-7	0.01	3.01	-7.0601	-6.01
3	-7	0.001	3.001	-7.006001	-6.001

(a) Complete the table. (5)

(b) Suggest the limit for the gradient of these chords as *h* tends to 0. (1)

b/ -6

36 A cuboid *ABCDEF* has width 2*x*, height *x* and depth *y*.

The volume of the cuboid is 600 cm³. The surface area of the cuboid is *S* cm².

(a) Show that $S = 4x^2 + \dfrac{1800}{x}$ **(5)**

(b) Determine the value of x for which the surface area of the cuboid is a minimum.   **(4)**

(c) Find, to the nearest integer, the minimum value of S.   **(1)**

---

a/   volume $= 2x \cdot x \cdot y$

$600 = 2x^2 y \longrightarrow y = \dfrac{600}{2x^2} = \dfrac{300}{x^2}$

$$S = 2(2x^2) + 2(2xy) + 2(xy)$$
$$= 4x^2 + 4xy + 2xy$$
$$= 4x^2 + 6xy$$

$$S = 4x^2 + 6x\left(\frac{300}{x^2}\right)$$

$$= 4x^2 + \frac{1800}{x}$$

b/ $\dfrac{dS}{dx} = 8x - 1800x^{-2}$

$$8x - \frac{1800}{x^2} = 0$$

$$8x^3 - 1800 = 0$$

$$x^3 = 225$$

$$x = 6.08 \text{ cm}$$

c/ $S = 4(6.08)^2 + \dfrac{1800}{6.08}$

$$= 444 \text{ cm}^2$$

37 (i) A curve has equation $y = 8x + \dfrac{1}{2x^2}$

(a) Find an expression for $\dfrac{dy}{dx}$ **(2)**

(b) Find an expression for $\dfrac{d^2y}{dx^2}$ **(2)**

(ii) Hence find the coordinates of the stationary point and determine its nature. **(5)**

$$y = 8x + \frac{1}{2}x^{-2}$$

a/ $\quad \dfrac{dy}{dx} = 8 - x^{-3}$

b/ $\quad \dfrac{d^2y}{dx^2} = 3x^{-4}$

ii/ $\quad 8 - x^{-3} = 0$

$$8 - \frac{1}{x^3} = 0$$

$$8 = \frac{1}{x^3}$$

$$x^3 = \frac{1}{8}$$

$$\underline{x = \frac{1}{2}}$$

$$y = 8\left(\frac{1}{2}\right) + \frac{1}{2\left(\frac{1}{2}\right)^2}$$

$$= 6$$

$\underline{\text{Stationary point at } \left(\frac{1}{2}, 6\right)}$

$$\dfrac{d^2y}{dx^2} = 3\left(\frac{1}{2}\right)^{-4} = 48$$

$\underline{+ve \quad \therefore \quad local \ minimum}$

38   Show that the only stationary point on the graph of $y = 2x^2 - 8\sqrt{x}$ is a minimum point at (1, -6)

$$y = 2x^2 - 8x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = 4x - 4x^{-\frac{1}{2}}$$

stationary point where $\frac{dy}{dx} = 0$

$$4x - 4x^{-\frac{1}{2}} = 0$$

$$x - x^{-\frac{1}{2}} = 0$$
$$x^{\frac{1}{2}}(x^{\frac{1}{2}} - 1) = 0$$

$$x^{\frac{1}{2}} = 0 \qquad x^{\frac{1}{2}} = 1$$

$$x = 0 \qquad \underline{\underline{x = 1}}$$

$\frac{dy}{dx}$ is undefined $\therefore$ only solution

at $x = 0$

$$4(0) - \frac{4}{0}$$

when $x = 1 \qquad y = 2(1)^2 - 8(\sqrt{1})$
$$= -6$$

$$\underline{\underline{(1, -6)}}$$

$$\frac{d^2y}{dx^2} = 4 + 2x^{-\frac{3}{2}}$$

when $x = 1 \qquad \frac{d^2y}{dx^2} = 6$

positive $\therefore$ $\underline{\text{MINIMUM}}$

39 Prove, from first principles, that the derivative of $4x$ is 4.

$$f(x) = 4x$$
$$f(x+h) = 4(x+h)$$

Differentiation

First Principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{4(x+h) - 4x}{h}$$

$$= \lim_{h \to 0} \frac{4x + 4h - 4x}{h}$$

$$= \lim_{h \to 0} \frac{4h}{h}$$

$$= \lim_{h \to 0} 4$$

$$= \underline{\underline{4}}$$

---

40 Prove, from first principles, that the derivative of x^3 is $3x^2$.

$$f(x) = x^3 \qquad f(x+h) = (x+h)^3$$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)(x^2 + 2xh + h^2) - x^3}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + hx^2 + 2x^2h + 2xh^2 + xh^2 + h^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= \lim_{h \to 0} 3x^2 + 3xh + h^2$$

$$= \underline{\underline{3x^2}}$$

41 Prove, from first principles, that the derivative of $2x^3$ is $6x^2$.

$$f(x) = 2x^3 \qquad f(x+h) = 2(x+h)^3$$
$$= 2(x+h)(x^2 + 2xh + h^2)$$
$$= 2(x^3 + x^2h + 2x^2h + 2xh^2 + xh^2 + h^3)$$
$$= 2(x^3 + 3x^2h + 3xh^2 + h^3)$$
$$= 2x^3 + 6x^2h + 6xh^2 + 2h^3$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{2x^3 + 6x^2h + 6xh^2 + 2h^3 - 2x^3}{h}$$

$$= \lim_{h \to 0} \frac{6x^2h + 6xh^2 + 2h^3}{h}$$

$$= \lim_{h \to 0} 6x^2 + 6xh + 2h^2$$

$$= \underline{\underline{6x^2}}$$

---

42 Prove, from first principles, that the derivative of $5x^2$ is $10x$.

$$f(x) = 5x^2 \qquad f(x+h) = 5(x+h)^2$$
$$= 5(x^2 + 2xh + h^2)$$
$$= 5x^2 + 10xh + 5h^2$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{5x^2 + 10xh + 5h^2 - 5x^2}{h}$$

$$= \lim_{h \to 0} \frac{10xh + 5h^2}{h}$$

$$= \lim_{h \to 0} 10x + 5h$$

$$= \underline{\underline{10x}}$$

43 Prove, from first principles, that the derivative of kx^3 is $3kx^2$.
Where k is a constant.

$$f(x) = kx^3 \quad f(x+h) = k(x+h)^3$$

$$= k(x+h)(x^2 + 2xh + h^2)$$

$$= k(x^3 + x^2h + 2x^2h + 2xh^2 + xh^2 + h^3)$$

$$= k(x^3 + 3x^2h + 3xh^2 + h^3)$$

$$= kx^3 + 3kx^2h + 3kxh^2 + kh^3$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{kx^3 + 3kx^2h + 3kxh^2 + kh^3 - kx^3}{h}$$

$$= \lim_{h \to 0} \frac{3kx^2h + 3kxh^2 + kh^3}{h}$$

$$= \lim_{h \to 0} \; 3kx^2 + 3kxh + kh^2$$

$$= \underline{\underline{3kx^2}}$$

44 A curve C has equation $y = 3x^2 + 1$

The point $P(3, 28)$ lies on the curve.

(a) Find the gradient of the tangent at P. **(2)**

The point Q with x-coordinate $(3 + h)$ also lies on C.

(b) Find the gradient of the line PQ, giving your answer in terms of h in its simplest form. **(3)**

(c) Explain briefly the relationship between part (b) and the answer to part (a). **(1)**

a/ $\dfrac{dy}{dx} = 6x$

when $x = 3$ $\dfrac{dy}{dx} = 6(3) = 18$

b/ $(3, 28)$ when $x = 3 + h$ x_2
x_1 y_1

$y = 3(3 + h)^2 + 1$
$= 3(9 + 6h + h^2) + 1$
$= 27 + 18h + 3h^2 + 1$
$= 3h^2 + 18h + 28$ y_2

$m = \dfrac{3h^2 + 18h + 28 - 28}{3 + h - 3}$

$= \dfrac{3h^2 + 18h}{h}$

$= 3h + 18$

c/ As h gets closer to zero the gradient gets closer to 18 (the answer to (a)).

45   Differentiate $3x^2 + x$ from first principles.

$$f(x) = 3x^2 + x \qquad f(x+h) = 3(x+h)^2 + (x+h)$$
$$= 3(x^2 + 2xh + h^2) + x + h$$
$$= 3x^2 + 6xh + 3h^2 + x + h$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 + x + h - (3x^2 + x)}{h}$$

$$= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 + x + h - 3x^2 - x}{h}$$

$$= \lim_{h \to 0} \frac{6xh + 3h^2 + h}{h}$$

$$= \lim_{h \to 0} \; 6x + 3h + 1$$

$$= \underline{\underline{6x + 1}}$$

46   Differentiate $4x - 3x^2$ from first principles.

$$f(x) = 4x - 3x^2 \qquad f(x+h) = 4(x+h) - 3(x+h)^2$$
$$= 4x + 4h - 3(x^2 + 2xh + h^2)$$
$$= 4x + 4h - 3x^2 - 6xh - 3h^2$$

$$f'(x) = \lim_{h \to 0} \frac{4x + 4h - 3x^2 - 6xh - 3h^2 - (4x - 3x^2)}{h}$$

$$= \lim_{h \to 0} \frac{4x + 4h - 3x^2 - 6xh - 3h^2 - 4x + 3x^2}{h}$$

$$= \lim_{h \to 0} \frac{4h - 6xh - 3h^2}{h}$$

$$= \lim_{h \to 0} 4 - 6x - 3h$$

$$= 4 - 6x$$

47 (a) Sketch the gradient function of the curve $y = x^3 + 3x^2 - 24x$ **(5)**

(b) Determine the set of values for which $x^3 + 3x^2 - 24x$ is increasing **(2)**

a/ $\dfrac{dy}{dx} = 3x^2 + 6x - 24$
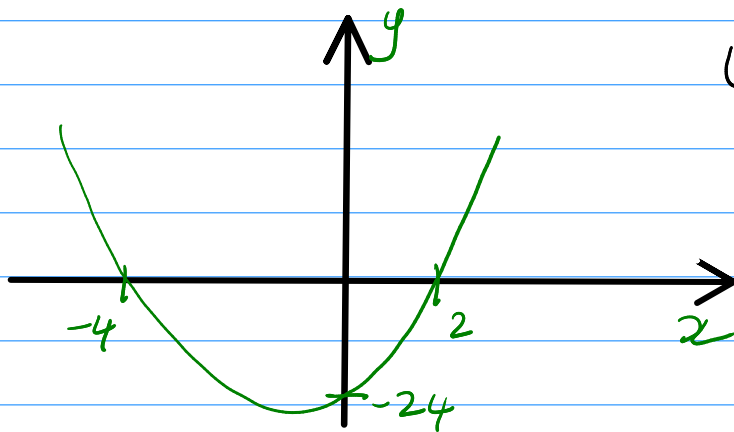
sketch $y = 3x^2 + 6x - 24$

crosses y when $x = 0$ $\quad y = -24$

crosses x when $y = 0$

$3x^2 + 6x - 24 = 0$

$x^2 + 2x - 8 = 0$

$(x + 4)(x - 2) = 0$

$x = -4 \quad x = 2$

b/ Increasing when $\quad x < -4 \quad$ or $\quad x > 2$

48 A curve has equation $y = x^3 + x^2$

A normal to the curve is drawn at the point where $x = 1$ and meets the x-axis at A and the y-axis at B.

(a) Find the area of OAB. **(6)**

(b) Use calculus to prove the curve has one local maximum and one local minimum point. **(6)**

a/ when $x = 1$ $\qquad y = (1)^3 + (1)^2$

$\quad x_1 \qquad\qquad = 2$

$\qquad\qquad\qquad\qquad y_1$

$\dfrac{dy}{dx} = 3x^2 + 2x$

when $x = 1 \qquad \dfrac{dy}{dx} = 5$

$\qquad\qquad\qquad\qquad M$

$y - 2 = 5(x - 1)$

$y - 2 = 5x - 5$

$\quad y = 5x - 3$

crosses y at -3 $\qquad$ crosses x when $\quad 0 = 5x - 3$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x = 3/5$

3/5

−3

$Area = \dfrac{1}{2} \cdot \dfrac{3}{5} \cdot 3$

$= \dfrac{9}{10} \text{ units}^2$

b/ $\quad \dfrac{dy}{dx} = 3x^2 + 2x$

turning points where $\dfrac{dy}{dx} = 0 \qquad 3x^2 + 2x = 0$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad x(3x + 2) = 0$

$\qquad\qquad\qquad\qquad\qquad x = 0 \qquad x = -\dfrac{2}{3}$

$\dfrac{d^2y}{dx^2} = 6x + 2 \qquad$ when $x = 0$ $\dfrac{d^2y}{dx^2} = 2$

$\qquad\qquad\qquad\qquad$ positive $\therefore$ local minimum

$\qquad\qquad\qquad$ when $x = -\dfrac{2}{3}$ $\dfrac{d^2y}{dx^2} = -2$

$\qquad\qquad\qquad$ negative $\therefore$ local maximum

49 The equation of a curve is $y = 4\sqrt{x} - 8x^2$

(a) Find $\dfrac{dy}{dx}$ (3)

(b) Find the coordinates of the turning point.   (3)

(c) Determine the nature of the turning point.   (2)

a/ $y = 4x^{\frac{1}{2}} - 8x^2$

$$\frac{dy}{dx} = 2x^{-\frac{1}{2}} - 16x$$

b/ $2x^{-\frac{1}{2}} - 16x = 0$

$$2x^{-\frac{1}{2}} = 16x$$

$$x^{-\frac{1}{2}} = 8x$$

$$1 = 8x^{\frac{3}{2}}$$

$$\frac{1}{8} = x^{\frac{3}{2}}$$

$$\frac{1}{64} = x^3$$

$$x = \frac{1}{4}$$

$$y = 4\sqrt{\frac{1}{4}} - 8\left(\frac{1}{4}\right)^2$$

$$= \frac{3}{2}$$

$$\left(\frac{1}{4}, \frac{3}{2}\right)$$

$$\frac{d^2y}{dx^2} = -x^{-\frac{3}{2}} - 16$$

when $x = \frac{1}{4}$ $\dfrac{d^2y}{dx^2} = -24$

negative $\therefore$ <u>Maximum</u>